ANALYTIC SOLUTIONS OF SELF-SIMILAR PROBLEMS OF PLANE TURBULENT GAS JETS OF VARIABLE COMPOSITION

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The initial and main regions of a plane turbulent gas jet of variable composition whose equation of state is given in general form are discussed.

Analytic methods of solving self-similar problems of turbulent jets of imperfect gases of constant composition are known [1, 2]. In [3] a numerical method is developed for solving the problem of a turbulent mixture of different ideal gases at substantially subsonic velocities, with the main attention being paid to the case of gases at the same temperature.

In the present article the equation of state is given in general form. Exact solutions are obtained for certain special cases of flow. The general case of a turbulent mixture of different imperfect gases is treated by an approximate analytic method.

1. We consider the initial (I) and main (II) regions of plane turbulent jets in a coflowing stream (Fig. 1). A gas whose parameters have the subscript 0 discharges into a medium moving with constant velocity. The distribution of all parameters at the lip of the nozzle is uniform. The parameters of the medium are constants with the subscript H. In general the gases are imperfect. We make van Driest's assumption that in addition to fluctuations of velocity, density, and enthalpy, there are fluctuations of the combinations (ρu) and (ρv). Prandtl's theory is used. Terms of molecular transport are neglected. The flow is isobaric.

We consider the initial region first. The origin of coordinates is placed at point A. Making the usual assumptions, the equations of continuity, motion, energy, concentration, and state for the initial region in dimensionless form are written as follows:

$$Z''' + \frac{3\vartheta'}{2\vartheta} Z'' + \left(\frac{\vartheta''}{2\vartheta} + bna \frac{\vartheta'}{2\vartheta} \right) Z' + b^3 Z + b^3 \frac{M}{1-M} = 0, \tag{1}$$

$$\tau'' + \left[\frac{(n-1)\vartheta'}{n\vartheta} + \frac{(n-2)Z''}{nZ'} \right] \tau' + \frac{u_0^2 (1-M)^2}{nh_0} (Z')^2 = 0, \tag{2}$$

$$\alpha'' + \left[\frac{(n-1)\vartheta'}{n\vartheta} + \frac{(n-2)Z''}{nZ'} \right] \alpha' = 0, \tag{3}$$

$$\vartheta = \Phi(\tau, \alpha). \tag{4}$$

Here $Z = (\langle u \rangle - u_H)/(u_0 - u_H)$, $\tau = \langle h \rangle / h_0$, $\vartheta = \langle \rho \rangle / \rho_0$, $\alpha = \langle \alpha \rangle$, $M = u_H/u_0$, and $n = 1/\mathrm{Pr}_T$. Primes denote differentiation with respect to η, $\eta = (\varphi - \varphi_2)/b$, $b = \varphi_1 - \varphi_2$, $\varphi = y/ax$, a is an empirical constant, and $n = l_h/l_u = l_\alpha/l_u = l_\rho/l_u$.

Using the boundary conditions

$$\begin{aligned} &\text{at } \eta = 1 \; Z = 1, \; Z' = 0, \; Z'' = b^2\varphi_1/(M-1), \; \tau = 1, \; \alpha = 1 \\ &\text{at } \eta = 0 \; Z = 0, \; Z' = 0, \; \tau = h_H/h_0 = N, \; \alpha = 0 \end{aligned} \tag{5}$$

and the integration constants, we determine the inner φ_1 and outer φ_2 boundaries of the jet. The flow in the main region can be considered as the discharge of a gas jet from a long in-

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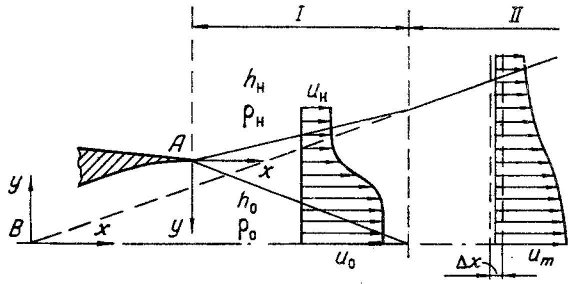

Fig. 1. Boundary layer of jet.

finitely narrow slot at point B into a co-flowing stream of gas of a different composition.

We divide the jet into small sections Δx. Following [2] we assume that in each section Δx the density and the ratio $M = u_H/u_m$ (the subscript m denotes parameters on the jet axis) are constants. We solve the problem for an arbitrary section Δx.

For substantially subsonic velocities the equations of motion, energy, concentration, and state have the form

$$2ZZ'Z'' + \frac{\vartheta'}{\vartheta} Z (Z')^2 - \frac{nab_1\vartheta'}{2\vartheta} Z^2 Z' - Z^3 b_1 - Z^2 \frac{b_1^3 M}{1-M} = 0, \tag{6}$$

$$Z'/Z = n\tau'/\tau, \tag{7}$$

$$Z'/Z = n\alpha'/\alpha, \tag{8}$$

$$\vartheta = \Phi_1(\varepsilon, \alpha). \tag{9}$$

Here $Z = (\langle u \rangle - u_н)/(u_m - u_н)$, $\tau = (\langle h \rangle - h_н)/(h_m - h_н)$, $\alpha = \langle \alpha \rangle / \alpha_m$, and $\vartheta = \langle \rho \rangle / \rho_m$. Primes denote differentiation with respect to η, $\eta = \varphi/b_1$, $b_1 = \varphi_1$.

The boundary conditions are

$$Z = 0 \quad \text{at} \quad \eta = 1; \quad Z = 1, \ Z' = 0, \ \tau = 1, \ \alpha = 1 \quad \text{at} \quad \eta = 0.$$

In the derivation of Eqs. (6)-(8) the following boundary conditions were used: at $\eta = 1$, $Z' = 0$, $\alpha = 0$.

2. We consider the problem for the initial region. We note first that as a result of integrating the energy and concentration equations the following relations can be obtained:

$$\begin{gathered} \tau = (1 - N)\alpha + N - I(1)\alpha + I(\eta), \ \alpha = F(\eta)/F(1), \\ I(\eta) = \int_0^\eta \left(\Theta^{-1} \int_0^\eta \beta\Theta d\eta\right) d\eta, \ F(\eta) = \int_0^\eta \Theta^{-1} d\eta, \\ \Theta = \vartheta^{(n-1)/n} (Z')^{(n-2)/n}, \ \beta = -(1 - M)^2 u_0^2 (Z')^2/(nh_0). \end{gathered} \tag{11}$$

We consider a special case which can be solved exactly. For low velocities when $u_0 \ll h_0$, and consequently $\beta = 0$, we obtain from (11)

$$\tau = (1 - N)\alpha + N. \tag{12}$$

Using (12), the equation of state (4) takes the form

$$\vartheta = \Phi((1 - N)\alpha + N, \ \alpha) = \bar{\Phi}(\alpha, \ N). \tag{13}$$

Suppose $Pr_T = 0.5$ ($n = 2$). In this case we obtain from (3)

$$\eta = \bar{F}(\alpha)/\bar{F}(1), \ \ \bar{F}(\alpha) = \int_0^\alpha \bar{\Phi}^{1/2} d\alpha. \tag{14}$$

Thus, Eqs. (12), (13), and (14) completely determine the profiles $\tau(\eta)$, $\alpha(\eta)$, and $\vartheta(\eta)$ for the initial region at low velocities and for $Pr_T = 0.5$. It should be noted that they are valid for an arbitrary equation of state.

In the equation of motion (1) we change variables from η to α. Using the fact that $d/d\eta = c_1\bar{\Phi}^{-1/2} d/d\alpha$, $c_1 = \bar{F}(1)$, we obtain

$$Z''' + \frac{nab\bar{\Phi}'}{2c_1\bar{\Phi}^{1/2}} Z' + \frac{b^3\bar{\Phi}^{3/2}}{c_1^3}\left(Z + \frac{M}{1-M}\right) = 0. \tag{15}$$

Here primes denote differentiation with respect to α.

Thus, the determination of the velocity profile in the case under consideration reduces to the integration of Eq. (15) with boundary conditions (5). The integration can be performed by Galerkin's method [2], for example.

3. We consider the solution of problem (6)-(10) for the main region. Integrating Eqs. (7) and (8) and satisfying boundary conditions (10), we obtain

$$\alpha = Z^{1/n}, \quad \tau = Z^{1/n}. \tag{16}$$

Substituting these expression into (9),

$$\vartheta = \Phi_1(Z^{1/n}, \ Z^{1/n}) = \bar{\Phi}(Z). \tag{17}$$

We introduce a dependent variable into the equation of motion (6) by the expression $p = Z'(\eta)$, and take Z as a new independent variable. Here $Z''(\eta) = p'(Z)p(Z)$. Using (17) we find $\vartheta'(\eta) = \bar{\Phi}'(Z)Z'(\eta)$. Since $\alpha \ll 1$, we neglect the third term in Eq. (6). Taking account of what has been said, the equation of motion takes the form

$$3p^2p' + 1.5(\bar{\Phi}'\bar{\Phi}^{-1} - Z^{-1})p^3 = 1.5b_1^3Z^2 + 1.5Mb_1^3Z/(1-M). \tag{18}$$

Here primes denote differentiation with respect to Z.

Equation (18) is linear in p^3. Its solution has the form

$$p^3 = (c_1 + \int_1^Z g(Z)\exp T dZ)\exp(-T), \tag{19}$$

$T(Z) = 1.5\int_1^Z (\bar{\Phi}^1\bar{\Phi}^{-1} - Z^{-1})dZ = 1.5\ln[\bar{\Phi}(Z)/Z]$ [in the integration use was made of the fact that $\Phi(1) = 1$], $g(Z) = 1.5b_1^3Z^2 + 1.5M_1^3Z/(1-M)$, $c_1 = \text{const}$.

Since $p = 0$ for $Z = 1$, $c_1 = 0$. Using (19) we now find

$$Z' = \sqrt[3]{\frac{3}{2}}\, b_1Q(Z),\ Q(Z) = \left[\frac{Z}{\bar{\Phi}(Z)}\right]^{1/2}\left\{\int_1^Z \frac{[\bar{\Phi}(Z)]^{3/2}}{Z^{1/2}}\left(Z + \frac{M}{1-M}\right)dZ\right\}^{1/3}. \tag{20}$$

Using the fact that $Z = 1$ for $\eta = 0$, we find

$$\eta = \frac{1}{b_1}\sqrt[3]{\frac{2}{3}}\int_1^Z Q^{-1}(Z)\,dZ.$$

Remembering that $Z = 0$ for $\eta = 1$, we find

$$b_1 = \sqrt[3]{\frac{2}{3}}\, Q^{-1}(Z)\,dZ.$$

Finally, we have

$$\eta = \int_1^Z Q^{-1}(Z)\,dZ/\int_1^0 Q(Z)\,dZ. \tag{21}$$

Equations (16), (17), and (21) completely determine the concentration, enthalpy, density, and velocity profiles in a cross section of the jet. The values of u_m, h_m, α_m, and ρ_m on the jet axis can be found by the method given in [2].

4. We consider an approximate analytic solution of problem (1)-(5) in the general case when no restrictions are imposed on $n = 1/Pr_T$, the function $\Phi(\tau, \alpha)$, and the flow velocity.

Using the equation of state (4) we calculate the value of the density ϑ at the boundaries of the jet: at $\eta = 1, \vartheta = \Phi(1, 1) = 1$; at $\eta = 0, \vartheta = \Phi(N, 0) = k$. We specify the velocity profile in the form

$$\vartheta = \tilde{\vartheta}(\eta, m) = (1 - k) E(\eta, m) + k, \tag{22}$$

where $E(\eta, m)$ is a certain family of functions depending on the parameter m, and $E(0, m) = 0$, $E(1, m) = 1$.

Using (22) and Eqs. (1)-(3) the profiles $Z(\eta)$, $\tau(\eta)$, and $\alpha(\eta)$ can be determined for each value of the parameter m. The velocity $Z(\eta)$ is determined from Eq. (1) by the Galerkin method [2], whereas the enthalpy $\tau(h)$ and the concentration $\alpha(\eta)$ are determined from (11). On the other hand, a correspondence can be established between each pair (α, τ) and the value of the density determined from Eq. (4). It is clear physically that if the values of ϑ determined from (22) and (4) are close to one another, the profiles found for $Z(\eta)$, $\tau(\eta)$, and $\alpha(\eta)$ will be close to the true profiles. Thus, we arrive at the following problem for the parameter m: It is required to determine the value of the parameter m for which the function $\tilde{\vartheta}(\eta, m)$ (22) is close to the function $\vartheta(\tau(\eta), \alpha(\eta))$ (4), e.g., in the sense of the relative error.

Taking account of the fact that the expressions $\tilde{\vartheta}(\eta, m)$ and $\vartheta(\tau(\eta), \alpha(\eta))$ coincide at the boundaries where $\eta = 1$, it can be required, e.g., that they agree also at some internal point $\eta = \eta_0$. As a particular family of functions $E(\eta, m)$ we can set $E(\eta, m) = \eta^m$.

5. We consider some results of the calculations for the initial region of a subsonic helium jet discharging into air ($\bar{R} = R_H/R_0 = 0.1382$, $\bar{c}_p = c_{pH}/c_{p0} = 0.1917$) for $Pr_T = 0.5$. In Fig. 2 the solid curves are the profiles of the concentration α and the density $\bar{\vartheta} = (\vartheta - k)/(1 - k)$, $k = \rho_H/\rho_0$ for $N = 0.1917$ and $N = 1$. The calculations were performed with the equations of Sec. 2. The enthalpy profile for these cases is easily obtained from Eq. (12). In Fig. 3 the solid curves for $N = 0.1917$ are the velocity profiles for two ratios of the velocities of miscible stream $M = 0$ and $M = 0.9$. Here the third approximations obtained by the Galerkin method are presented. Calculations showed that the difference between the third approximation and the next is 0.1% in the cases considered.

To estimate the accuracy of the approximate solutions this same problem for $N = 0.1917$ was solved by the method presented above. The results of the dashed curves in Figs. 2 and 3. The value $\eta_0 = 0.325$ chosen corresponds to $m = 0.3172$. It is clear that the agreement of the approximate and exact solutions is satisfactory.

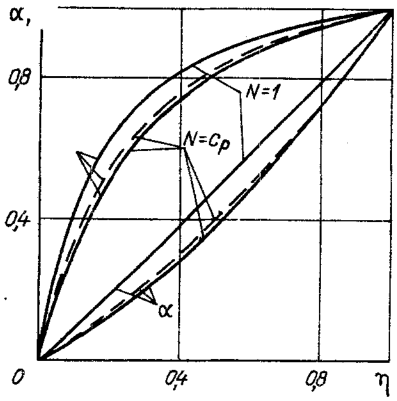

Fig. 2. Profiles of concentration α and density $\bar{\vartheta}$.

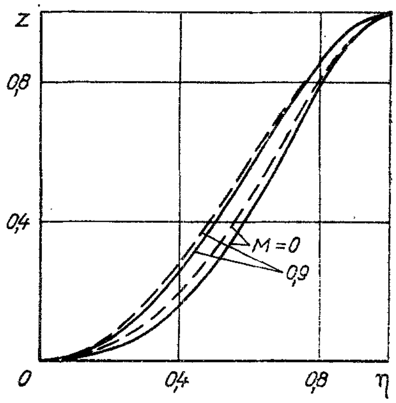

Fig. 3. Velocity profiles.

NOTATION

x, y, Cartesian coordinates; η, transformed coordinate; u, v, velocity components; h, enthalpy; ρ, density; α, concentration; Z, relative velocity; τ, ε, relative enthalpy; ϑ, relative density; φ_1, φ_2, boundaries of jet; l_u, l_h, l_ρ, l_α, mixing lengths; a, mixing-length constant; c_p, specific heat at constant pressure; R, gas constant; c_1, integration constant. The subscript H refers to the outer and 0 to the inner boundary; < >, average value.

LITERATURE CITED

1. V. A. Golubev, in: Turbulent Jets of Air, Plasma, and Real Gas, Plenum Publ. (1969).
2. V. I. Bakulev, in: Turbulent Jets of Air, Plasma, and Real Gas, Plenum Publ. (1969).
3. G. N. Abramovich, S. Yu. Krasheninnikov, A. N. Sekundov, and I. P. Smirnova (Editors of Mathematical Physics Literature), Turbulent Mixture of Gas Jets [in Russian], Nauka, Moscow (1974).

A FLAT JET IN A GRANULAR BED OF FINITE HEIGHT

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The distribution of gas flows in the vicinity of the jet is discussed and the conditions of disruption of the static equilibrium of the bed, the formation and growth of a cavity, and the jet breakthrough of the bed are investigated qualitatively.

In the blowing of gas into a granular bed through an opening in its base one can distinguish three main modes of spread of the gas jet differing in a qualitative respect [1]. At low flow rates ordinary gas filtration occurs without disruption of the continuity of the bed. With an increase in the flow rate above some critical value, depending on the physical parameters of the particles and the gas, the size of the opening, and the geometry of the bed, the initial static equilibrium of the bed is disrupted: Near the orifice of the jet a cavity forms with a relatively small number of particles circulating in it, surrounded by granular material which is motionless as before, whose size grows with a further increase in the flow rate. When the flow rate exceeds some new critical value the equilibrium of the bed with the cavity also becomes impossible: The jet "breaks through" the bed with the establishment of a steady regime of the type studied in [2].

The pattern described also reflects the initial stages of development of a fountain in spouting beds, discussed in [3, 4], for example. The formation and growth of the cavity are analogous to the appearance and development of the "initial channel" in the mode of "internal" spouting [5, 6], as well as to the partially fluidized lower part of a granular bed in trough and conical apparatus [6-8]. The jet breakthrough of a bed is equivalent to this initial channel breaking through the entire bed and to the onset of a true spouting mode. In this case the critical values of the flow rate mentioned above correspond to the first two critical spouting velocities introduced in [9].

The theoretical analysis of the spread of a jet in a granular bed with disruption of the continuity of the latter and the analysis of the conditions of interchange of the above-indicated modes require the joint solution of two very complicated problems with an unknown boundary. First of all, we need to find the distribution of the hydraulic forces acting on the stationary granular material in the vicinity of the cavity on the part of the filtering gas in it, for which we must solve the filtration problem (nonlinear in the general case) on the distribution of gas flows in the bed. In addition, we must study the static equilibrium for a given distribution of hydraulic forces, and the unknown shape of the cavity must, in principle, be determined in the course of such an investigation. A rigorous solution of these interconnected problems is scarcely possible at present. Therefore, they are analyzed below with a number of simplifying assumptions only for a plane bed unbounded in a horizontal

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